Fermi golden rule derivation

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I. THE PROBLEM

Consider a discrete level ("quantum dot") coupled to a continuous band ("lead"):

$$\mathcal{H} = \epsilon_0 |d\rangle\langle d| + \sum_k \epsilon_k |k\rangle\langle k| + \sum_k \left[V_k |k\rangle\langle d| + V_k^* |d\rangle\langle k| \right] \tag{1.1}$$

The basis states are normalized to 1 adn orthogonal to each other. The sum over the continuous spectrum should be understood in the sense of an integral,

$$\sum_k F(\epsilon_k) \to \int \rho(\omega)\, F(\omega)\, d\omega \tag{1.2}$$

where $\rho(\omega)$ is the density of states. The number of states with energies $\epsilon_k \in [\omega..\omega + d\omega]$ is $\rho(\omega)\, d\omega$. This number is very large, and diverges as the linear size of the lead goes to infinity.

We shall solve perturbatively the Schrodinger equation

$$i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle = \mathcal{H} |\psi(t)\rangle \tag{1.3}$$

subject to initial condition:

$$|\psi(t = 0)\rangle = |d\rangle \tag{1.4}$$

The perturbation is the tunneling term, proportional to V_k.

Let us parameterize the state vector:

$$|\psi\rangle = a_d |d\rangle + \sum_k a_k |k\rangle \tag{1.5}$$

The initial condition

$$a_d = 1\,, \quad a_k = 0 \tag{1.6}$$

Substituting Eqs.(1.5) into (1.3) and equating the corresponding coefficients in front of the (linear independnet) basis vectors gives:

$$i\hbar\, \dot{a}_d = \epsilon_0\, a_d + \sum_k V_k\, a_k \tag{1.7}$$

$$i\hbar\, \dot{a}_k = V_k^*\, a_d + \epsilon_k\, a_k \tag{1.8}$$

$$\tag{1.9}$$

We shall be interested in the probability to remain in state $|d\rangle$:

$$P(t) = |\, \langle d|\psi(t)\rangle \,|^2 = |a_d(t)|^2 \tag{1.10}$$

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A. Perturbative solution (Fermi golden rule)

Order-by-order expansion of the state amplitudes is

$$a_d(t) = a_d^{(0)}(t) + a_d^{(1)}(t) + a_d^{(2)}(t) + \ldots \tag{1.11}$$
$$a_k(t) = a_k^{(0)}(t) + a_k^{(1)}(t) + a_k^{(2)}(t) + \ldots \tag{1.12}$$
$$\tag{1.13}$$

Substituting (1.11) into (1.7) and taking into account the initial condition gives

$$\underline{\text{order 0:}} \quad i\hbar\, \dot{a}_d^{(0)}(t) = \epsilon_0 a_d^{(0)}(t) \tag{1.14}$$
$$i\hbar\, \dot{a}_k^{(0)}(t) = \epsilon_k a_k^{(0)}(t) \tag{1.15}$$
$$a_d^{(0)}(t) = e^{-i\epsilon_0 t/\hbar} \tag{1.16}$$
$$a_k^{(0)}(t) = 0 \tag{1.17}$$
$$\tag{1.18}$$

First order:

$$\underline{\text{order 1:}} \quad i\hbar\, \dot{a}_d^{(1)}(t) = \epsilon_0 a_d^{(1)}(t) + \sum_k V_k \underbrace{a_k^{(0)}}_{0} = \epsilon_0 a_d^{(1)}(t) \tag{1.19}$$
$$a_d^{(1)}(t) = 0 \tag{1.20}$$
$$i\hbar\, \dot{a}_k^{(1)}(t) = \epsilon_k a_k^{(1)}(t) + V_k^* \underbrace{a_d^{(0)}}_{e^{-i\epsilon_0 t/\hbar}} \tag{1.21}$$
$$a_k^{(1)}(t) = \frac{V_k^*}{\epsilon_0 - \epsilon_k}\left[e^{-i\epsilon_0 t/\hbar} - e^{-i\epsilon_k t/\hbar}\right] \tag{1.22}$$

Second order:

$$\underline{\text{order 2:}} \quad i\hbar\, \dot{a}_d^{(2)}(t) = \epsilon_0 a_d^{(2)}(t) + \sum_k \frac{V_k V_k^*}{\epsilon_0 - \epsilon_k}\left[e^{-i\epsilon_0 t/\hbar} - e^{-i\epsilon_k t/\hbar}\right] \tag{1.23}$$