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Physica A 377 (2007) 115–124
PHYSICA A
www.elsevier.com/locate/physa

Anderson localization problem: An exact solution for 2-D anisotropic systems

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Received 5 March 2006; received in revised form 29 October 2006
Available online 4 December 2006

Abstract

Our previous results [V.N. Kuzovkov, W. von Niessen, V. Kashcheyevs, O. Hein, J. Phys. Condens. Matter 14 (2002) 13777] dealing with the analytical solution of the two-dimensional (2-D) Anderson localization problem due to disorder is generalized for anisotropic systems (two different hopping matrix elements in transverse directions). We discuss the mathematical nature of the metal–insulator phase transition which occurs in the 2-D case, in contrast to the 1-D case, where such a phase transition does not occur. In anisotropic systems two localization lengths arise instead of only one length.

Keywords: Random systems; Anderson localization; Phase diagram

1. Introduction

Anderson localization [1] remains one of the main problems in the physics of disordered systems (see e.g., the review articles [2–4]). In the series of our previous papers [5–7] we presented an exact analytic solution to this problem. By the exact solution we mean the calculation of the phase diagram for the metal–insulator system. We have been able to solve the two dimensional (2-D) problem [5]. We have shown then that the phase of delocalized states exists for a non-interacting electron system. The main aim of the paper [6] was the generalization of the results to the case of higher dimensional spaces (N-D). In Ref. [7] we discussed the mathematical details of the new analytical approach for calculating the phase diagram. An exact solution is only possible for the conventional Anderson model: the tight-binding approximation with diagonal disorder, where on-site potentials are independently and identically distributed.

It is well known that the exact results in the field of phase transitions (the metal–insulator transition is a particular case of a phase transition) are exceedingly rare [8,9]. This is why any extension of the applicability

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doi:10.1016/j.physa.2006.11.005

range of analytical methods in this field is of great interest. In this paper we extend our approach [5–7,10] to *anisotropic* media [11–14], but else remaining in the framework of a conventional Anderson model.

Incorporation of an anisotropy into the tight-binding approximation with diagonal disorder is also methodologically valuable. Before the exact solution was obtained for $D>1$ [5–7], analytical methods concentrated on the $D=1$ case [15,16]. However, the specific topology of 1-D systems does not permit the extension of the results to higher dimensions. This is true in particular for the Ising model, where an exact analytical solution for the 1-D case has nothing to do with the 2-D solution obtained by Onsager [8,9]. As it is well known for the Anderson localization problem, all states in the 1-D system are localized (i.e., there is no metal–insulator transition). This is a particular result of a general theory [9] that no phase transitions are possible in 1-D systems with short-range interactions. In fact, phase transitions (e.g., in the Ising model [8,9] or the Anderson localization problem [5]) are observed only starting with $D=2$.

It should be realized that the approximate methods are also of a limited use here. In particular, traditional perturbation theory works perfectly in the 1-D case [15], which is well demonstrated by the analysis of the exact solution in Refs. [5,17]. Random potentials can be treated in the 1-D Anderson localization problem as a small parameter, which is used in a series expansions of physical quantities. However, this approach fails [9] for systems with phase transitions in $D>1$, since physical quantities here are no longer described by analytical functions and series expansions. The same is valid for the Lyapunov exponent γ (which is the inverse of the localization length ξ in the Anderson problem [5]).

Of particular interest is the understanding of the mathematical nature of the phase transition: how does the analytical character of the exact solution for the 1-D problem change to a non-analytical character of the exact solution for the 2-D case? For the *discrete* spatial dimensions (1-D or 2-D) a simple comparison of the two relevant solutions does not help us. However, a "*continuous*" treatment of the spatial dimension for anisotropic systems, performed in this paper, provides much more insight. We have put the word continuous within quotation marks because we have in the present case of anisotropic systems an interesting possibility of a transition from a 2-D system to a 1-D system, by considering the limit $\kappa \to 0$ for parameter of anisotropy κ.

The paper is organized as follows. In Section 2 we discuss the basic equations for the isotropic problem [5], which are generalized there for the anisotropic case. Section 3 presents the main results for the anisotropic problem. We demonstrate how the study of the limiting case of a strong anisotropy permits us to establish a relation between 1-D and 2-D cases.

2. Recursion relation and the filter function

2.1. Isotropic system

Let us start with the Schrödinger equation for the isotropic system (the lattice constant and the hopping matrix element are set equal to unity)

$$\psi_{n+1,m}+\psi_{n-1,m}+\psi_{n,m+1}+\psi_{n,m-1}=(E-\varepsilon_{n,m})\psi_{n,m}. \tag{1}$$

The on-site potentials $\varepsilon_{n,m}$ are independently and identically distributed with existing first two moments, $\langle\varepsilon_{n,m}\rangle=0$ and $\langle\varepsilon_{n,m}^2\rangle=\sigma^2$.

Eq. (1) can be written as the recursion

$$\psi_{n+1,m}=-\varepsilon_{n,m}\psi_{n,m}-\psi_{n-1,m}+\mathscr{L}\psi_{n,m}, \tag{2}$$

where the operator $\mathscr{L}$ acts on the index m and is defined by the relation

$$\mathscr{L}\psi_{n,m}\equiv E\psi_{n,m}-\sum_{m'=\pm1}\psi_{n,m+m'}. \tag{3}$$

The standard initial (boundary) conditions are $\psi_{0,m}=0$ and $\psi_{1,m}=\alpha_m$, respectively.

The existence of some fundamental 1-D numerical series h_n ($n=0,1,\ldots,\infty$), the so-called system function or *filter*, was proved rigorously in Refs. [5–7]. A study of the asymptotic behaviour of this series allows to define uniquely the *phase diagram* of the system. Namely, the series h_n is bounded

$$|h_n|<\infty, \tag{4}$$

for the delocalized states, but increases without bound

$$\lim_{n\to\infty} |h_n| \to \infty, \tag{5}$$

for the localized states. The actual numerical values are not important in this respect. The determination of the existence region of the bounded series, Eq. (4), permits to find also that for the delocalized states, i.e., to obtain the system's phase diagram. In its turn, the asymptotics of the series (5), $h_n \propto \exp(2\gamma n)$, for the localized states allows us to extract the *Lyapunov exponent* γ, or the localization length $\xi = 1/\gamma$.

It is more convenient to determine the phase diagram and the Lyapunov exponent γ by using the complex variable z (Z-transform)

$$H(z) = \sum_{n=0}^{\infty} \frac{h_n}{z^n}. \tag{6}$$

In this case, the physical problem of the localized/delocalized states is reduced to the mathematical problem of the search for the poles of the complex variable function $H(z)$ [5–7], which is also called the *filter*. Eq. (5) means in this case that $H(z)$ has the pole at $z = \exp(2\gamma)$.

For the isotropic system the filter can be found exactly [5–7]:

$$H^{-1}(z) = 1 - \frac{\sigma^2}{2\pi}\frac{z+1}{z-1}\int_{-\pi}^{\pi} \frac{\mathrm{d}k}{w^2 - \mathscr{L}^2(k)}, \tag{7}$$

$$w^2 = \frac{(z+1)^2}{z}, \tag{8}$$

where

$$\mathscr{L}(k) = E - 2\cos(k) \tag{9}$$

corresponds to the operator $\mathscr{L}$ and arises due to use of the Fourier transform.

Note that solution of the Schrödinger equation (2) depends on chosen initial conditions, $\psi_{0,m} = 0$ and $\psi_{1,m} = \alpha_m$. In contrast, as follows from the definition, Eq. (7), the filter $H(z)$ is independent of the initial conditions (i.e., the field variables α_m), and thus is *not* the functional of the wave function $\psi_{n,m}$. Following Refs. [5–7], let us briefly explain the mathematical nature of the filter $H(z)$. According to the definition, Eq. (3), the operator $\mathscr{L}$ acts on the wave functions. Expression (7) is a functional of $\mathscr{L}$ and a disorder parameter σ. The analysis [5–7] shows that the function $H(z)$ is also an *operator* which acts, however, not on wave functions but on certain mathematical objects called *signals*. Only these signals are functionals of the wave functions.

The relation between the filter $H(z)$ and signals (input signal $S^{(0)}(z)$ and output signal $S(z)$),

$$S(z) = H(z)S^{(0)}(z), \tag{10}$$

$$S^{(0)}(z) = \frac{1}{2\pi}\frac{z+1}{z-1}\int_{-\pi}^{\pi} \frac{|\alpha(k)|^2\,\mathrm{d}k}{w^2 - \mathscr{L}^2(k)}, \tag{11}$$

has a certain similarity with the transition to the operator formalism in quantum mechanics. Note that the boundary conditions (field α_m or Fourier transform $\alpha(k)$) influence only functions $S^{(0)}(z)$ which are independent of the parameter σ. In other words, functions $S^{(0)}(z)$ correspond to solutions in a *completely ordered system*. Introduction of disorder, $\sigma > 0$, transforms the initial solution $S^{(0)}(z)$ into $S(z)$, where the operator $H(z)$ describes this transformation.

The concept of the filter function is a general and abstract description of the problem of localization. Instead of analyzing wave functions, i.e., different signals, it is sufficient to analyze properties of the fundamental *localization operator* $H(z)$ by means of the theory for functions of complex variables. The signal concept (input and output signals) and the filter function are key elements in the theory of signals used earlier [5–7]. In these papers the connection between the Anderson localization problem and signal theory is discussed and the most important concept of the proposed method, the filter $H(z)$, is defined. The knowledge of papers [5–7] presenting the main formalism is prerequisite for understanding the present paper.

Let us discuss Eq. (5). The localized states correspond to values of $\gamma > 0$, i.e., a divergence of the filter function h_n (or signals). This divergence is mathematically similar to the divergence of averages in the diffusion motion [7]. That is, transition from delocalized to localized states can be treated as a *generalized diffusion* with a noise-induced first-order phase transition. The generalized diffusion arises due to the instability of a fundamental mode corresponding to the averages of wave functions (correlators and signals).

2.2. Anisotropic system

The Schrödinger equation for the 2-D anisotropic medium

$$t_1(\psi_{n+1,m} + \psi_{n-1,m}) + t_2(\psi_{n,m+1} + \psi_{n,m-1}) = (E - \varepsilon_{n,m})\psi_{n,m} \tag{12}$$

contains two hopping matrix elements: t_1, t_2. Let us use the units, where $\max\{t_1, t_2\} = 1$, and $\min\{t_1, t_2\} = \kappa$ with $0 \leqslant \kappa \leqslant 1$. The parameter κ characterizes the system's anisotropy. For example, the bandwidth is $|E| \leqslant E_{max} = 2(1 + \kappa)$.

Eq. (12) can also be written as a recursion relation. Let us assume $\varepsilon'_{n,m} = \varepsilon_{n,m}/t_1$ and define the operator $\mathscr{L}'$ through the relation

$$\mathscr{L}'\psi_{n,m} \equiv \frac{E}{t_1}\psi_{n,m} - \frac{t_2}{t_1}\sum_{m'=\pm 1}\psi_{n,m+m'}. \tag{13}$$

As a result, we obtain the recursion

$$\psi_{n+1,m} = -\varepsilon'_{n,m}\psi_{n,m} - \psi_{n-1,m} + \mathscr{L}'\psi_{n,m}, \tag{14}$$

functionally similar to Eq. (2). The equation for the filter function $H(z)$ transforms, respectively, to

$$H^{-1}(z) = 1 - \frac{(\sigma')^2}{2\pi}\frac{z+1}{z-1}\int_{-\pi}^{\pi}\frac{\mathrm{d}k}{w^2 - (\mathscr{L}'(k))^2}, \tag{15}$$

where $(\sigma')^2 = \langle(\varepsilon'_{n,m})^2\rangle = \sigma^2/t_1^2$,

$$\mathscr{L}'(k) = \frac{E}{t_1} - \frac{t_2}{t_1}2\cos(k). \tag{16}$$

In an anisotropic system there are two distinct directions. We give them different names in order to distinguish them later. The choice $t_1 = 1$, $t_2 = \kappa$ corresponds to the direction of strong binding, while $t_1 = \kappa$, $t_2 = 1$ to the direction of weak binding. Note that the former case (strong binding) should only be considered for establishing the correspondence with the exact 1-D solution [5,17] in the limit $\kappa \to 0$. Indeed, for $\kappa = 0$ one gets $t_1 = 1$, $t_2 = 0$ and the operator $\mathscr{L}' = E$. Eq. (14) transforms, respectively, into the ensemble of similar equations characterized by a neutral index m,

$$\psi_{n+1,m} = -\varepsilon_{n,m}\psi_{n,m} - \psi_{n-1,m} + E\psi_{n,m}, \tag{17}$$

each of which is equivalent to the 1-D equation

$$\psi_{n+1} = -\varepsilon_n\psi_n - \psi_{n-1} + E\psi_n. \tag{18}$$

2.3. Useful relation

Let us use the relation

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\mathrm{d}k}{\zeta \pm 2\cos(k)} = \frac{1}{\mathrm{i}\sqrt{4 - \zeta^2}}, \tag{19}$$

which holds provided the imaginary part of the complex variable ζ is non-negative, $\mathrm{Im}\,\zeta \geqslant 0$ (ζ is defined in the upper semi-plane).

The integrand (15) can be presented as

$$\frac{1}{w^2-(\mathscr{L}'(k))^2}=\frac{1}{2w}\left[\frac{1}{w-\mathscr{L}'(k)}+\frac{1}{w+\mathscr{L}'(k)}\right], \tag{20}$$

and then Eq. (19) can be used, provided that the parameter w is also defined in the upper semi-plane, $\mathrm{Im}\, w \geqslant 0$.

The transformation from the complex variable z to w (*conformal mapping*) gives a parametric presentation of the filter function $H(z) \equiv \mathscr{H}(w)$ [6] convenient for analysis. Since the relation between the variables z and w is *non-linear*, this relation is not *unique*.

2.4. Conform transformation and phase index P

If the complex variable $w = u + iv$ is defined in the upper semi-plane, $v \geqslant 0$, the relation

$$w = P(z^{1/2} + z^{-1/2}) \tag{21}$$

is valid [5], where the parameter $P = \pm 1$. As shown below, P serves as an index numbering different-type solutions: localized ($P = +1$) and delocalized ($P = -1$), i.e., it characterizes the phase state of a system.

For $P = +1$ Eq. (21) characterizes the conform transformation of the outer region of the unit circle, $|z| \geqslant 1$, onto the upper w-semi-plane. Similarly, for $P = -1$ the conform transformation of the inner part of the unit circle, $|z| \leqslant 1$, occurs.

The inverse transformation is described by the relations

$$z = -1 + \frac{w^2}{2} + P\frac{wi}{2}\sqrt{4-w^2}, \tag{22}$$

provided

$$\frac{z+1}{z-1} = P\frac{w}{i\sqrt{4-w^2}}. \tag{23}$$

2.5. Anderson localization and first-order phase transition

As was shown [5,6], for arbitrary dimension D the phase of localized states (the solution with $P = +1$) is found for an arbitrary disorder σ. In contrast, the delocalized phase ($P = -1$) arises only for $D \geqslant 2$. It exists within the limited energy band and below the critical disorder parameter σ.

A comparison of the 1-D and 2-D cases shows their qualitative differences. In 1-D even infinitesimal disorder destroys the localized states; their existence range degenerates into a single point $\sigma \equiv 0$ [5,17]. Similarly to many other problems of phase transitions [8], the (metal–insulator) phase transition in 1-D is impossible. In this respect, the delocalized states at $\sigma \equiv 0$, which are unstable under any perturbation, cannot be treated as a special phase. It is more reasonable to treat these states as the limiting case of the localized states which only exist for 1-D. As a consequence, the physical characteristics of the single-phase system such as the Lyapunov exponents, $\gamma = \gamma(\sigma, E)$, are *analytical functions* of the energy E and the disorder parameter σ. Moreover, the perturbation theory also applies here: $\gamma = \gamma(\sigma, E)$ can be expanded in a series of a small parameter σ^2 [5]. The 1-D system is characterized by a *continuum* transition due to the lack of disorder, $\sigma \to 0$. The limiting value $\lim_{\sigma\to 0}\gamma(\sigma, E) = \gamma(0, E)$ has obviously to coincide with that for the delocalized states, $\gamma = 0$. In other words,

$$\gamma(0, E) = 0 \tag{24}$$

holds.

The 2-D case differs *qualitatively* from the 1-D case. First of all, in the existence range of the second solution ($P = -1$) the first solution ($P = +1$) is also always present. In other words, two phases *co-exist* [5,6]. Therefore, the metal–insulator transition has to be interpreted as a *first-order* phase transition. Two fundamental points should be stressed here. (i) In any system with dimension larger or equal to two and with a phase transition, the physical properties are no longer analytical functions of their variables. That is, such

properties as the Lyapunov exponents $\gamma(\sigma, E)$ can manifest peculiarities, e.g., step-like changes, and thus the expansion in *small parameter* σ^2 no longer holds here [5]. (ii) The first-order phase transitions reveal additional peculiarities. The co-existing phases (in our case, the delocalized states with $\gamma = 0$ and localized states with $\gamma(\sigma, E) \neq 0$) have nothing in common, and thus there is no reason to expect that Eq. (24) holds. That is, in a system with first-order phase transition,

$$\gamma(0, E) \neq 0, \tag{25}$$

which illustrates once more the above-discussed non-analytical behaviour of the physical properties.

As was recently noted [5], $\gamma(\sigma, E)$ in the Anderson localization problem can be treated as the *long-range* order parameter. Indeed, the two different phases reveal different γ parameters, $\gamma \equiv 0$ and $\gamma \neq 0$, respectively. The first-order transition is always characterized by the step-like change of the long-range parameter [9]. In this respect, Eq. (25), being the particular case of a more general relation $\gamma \neq 0$ for the phase of delocalized states, seems quite self-evident, despite possible exceptions discussed below.

Therefore, the analysis of the limiting transition to 1-D in the anisotropic system permits, on the one hand, to understand the mechanism for the change from non-analytical to analytical solutions for the physical properties, and, on the other hand, gives an additional check of our results [5,6] since the difference between Eqs. (24) and (25) is non-trivial.

3. Results

3.1. General remarks

Using the above-obtained relations, one gets the following parametric presentation of the filter function:

$$\frac{1}{\mathscr{H}(w)} = 1 + P\frac{\sigma^2}{2t_1 t_2\sqrt{4 - w^2}} \sum_{v=\pm 1} \frac{1}{\sqrt{4 - Q_v^2(w, E)}}, \tag{26}$$

where

$$Q_v(w, E) = \frac{t_1}{t_2} w + v\frac{E}{t_2}. \tag{27}$$

Let us consider now the properties of the filter function $\mathscr{H}(w) = \mathscr{H}_-(w)$ at $P = -1$. As was shown in Refs. [5,6], the filter analysis permits to obtain the phase diagram, i.e., to determine the existence range of the delocalized states. For this one has to determine the poles of $H(z)$, i.e., to find the roots of the equation $H^{-1}(z) = 0$. In the parametric presentation for $P = -1$, we seek roots of

$$\mathscr{H}_-^{-1}(w) = 0, \tag{28}$$

or

$$\frac{\sigma^2}{2t_1 t_2\sqrt{4 - w^2}} \sum_{v=\pm 1} \frac{1}{\sqrt{4 - Q_v^2(w, E)}} = 1. \tag{29}$$

The peculiarity of the case $P = -1$ [5,6] is that its physical interpretation is possible either in the *absence* of the poles (which is the case for $D > 2$ [6]) or if the poles lie on the unit circle $|z| = 1$ (*marginal stability*), i.e.,

$$z = \exp(\pm i2\varphi) \tag{30}$$

(pairs of complex values), $\varphi \in (0, \pi)$. The latter case applies to 2-D isotropic systems [5]. It can also be shown using the method in Ref. [6] that taking into account the anisotropy does not affect Eq. (30) (we omit here the proof). As a result, in the general case of an anisotropic system, it follows from Eq. (21) that

$$w = -2\cos(\varphi), \tag{31}$$

i.e., the w parameter is *real*. This obviously simplifies the solution of Eq. (29).

The equation

$$\mathscr{H}_{-}^{-1}(w=0)=0 \tag{32}$$

determines the threshold disorder $\sigma=\sigma_0(E)$ which defines the limits of existence of the delocalized phase [5,6].

It is easy to conclude that the real roots w of Eq. (28) satisfy the condition $|Q_v(w,E)|\leqslant 2$, which is equivalent to

$$t_1|w|+|E|\leqslant 2t_2. \tag{33}$$

Simultaneously, $0\leqslant|w|\leqslant 2$ has to hold.

Consider now the filter function $\mathscr{H}_{+}(w)$ for $P=+1$. This function has always the only pole [5]

$$z=\exp(2\gamma), \tag{34}$$

which lies on the real axis with $\gamma\geqslant 0$, where γ is interpreted as the Lyapunov exponent. In the parametric presentation the parameter w is also real:

$$w=2\cosh(\gamma). \tag{35}$$

To find the Lyapunov exponent, one has to determine the root of the equation

$$\mathscr{H}_{+}^{-1}(w)=0, \tag{36}$$

or

$$\frac{\sigma^2}{2t_1t_2\sqrt{w^2-4}}\left[\frac{1}{\sqrt{Q_{+}^2(w,E)-4}}+\frac{1}{\sqrt{Q_{-}^2(w,E)-4}}\right]=1. \tag{37}$$

The parameter w sought for satisfies the natural condition of positive arguments of all roots of Eq. (37), so the simple condition

$$t_1w-|E|\geqslant 2t_2 \tag{38}$$

has to be fulfilled, along with $w\geqslant 2$.

3.2. Isotropic system

Let us summarize for further analysis the main results for an isotropic system.

3.2.1. The case of $P=-1$ (delocalized states)

In an isotropic system, the real roots w have to fulfill $0\leqslant|w|\leqslant 2$ [5]. This is possible only in the limited energy range $|E|\leqslant E_0=2$, otherwise one obtains complex roots with an unclear interpretation. That is, the delocalized states exist only within a certain energy range around the zone centre. In this energy range, an increase of the disorder parameter σ shifts the poles in the parametric presentation from $|w|=w_0=2-|E|\leqslant 2$ towards the limiting value of $w=0$. For further disorder increases, the real roots disappear and thus the formal solutions of Eq. (28) permit no physical interpretation.

Using Eq. (32) at $t_1=t_2=1$, one gets [5]

$$\sigma_0(E)=2(1-E^2/E_0^2)^{1/4}. \tag{39}$$

3.2.2. The case $P=+1$ (localized states)

Eq. (38) determines the existence of real roots $w\geqslant u_0=2+|E|$ of Eq. (37) [5,6]. Eq. (35) holds here with $\gamma=\gamma(\sigma,E)$. In the parametric presentation, the value of $w=u_0$ corresponds to the absence of disorder, $\sigma=0$. In this case,

$$\gamma(0,E)=\sinh^{-1}\left(1+\frac{|E|}{2}\right). \tag{40}$$

This result confirms Eq. (25). The continuity, Eq. (24), is found only at the band centre, $E = 0$, when $\gamma(0, 0) = 0$. In this case, one obtains for the Lyapunov exponent

$$\gamma(\sigma, 0) = \sinh^{-1}\left(\frac{\sigma}{2}\right). \tag{41}$$

E.g., for $\sigma \to 0$ one has from Eq. (41) $\gamma(\sigma, 0) \propto \sigma$. It follows from this that the function $\gamma(\sigma, E)$ cannot be represented as a series in powers of σ^2, i.e., perturbation theory is not applicable to the 2-D Anderson problem.

3.3. Strong binding direction

Let us now assume that $t_1 = 1$, $t_2 = \kappa$. As was mentioned in Section 2.2, this case in the limit of $\kappa \to 0$ corresponds to the transition to the 1-D case.

3.3.1. The case $P = -1$ (delocalized states)

In this case, Eq. (33) transforms into

$$|w| + |E| \leqslant 2\kappa. \tag{42}$$

Along with the relation $0 \leqslant |w| \leqslant 2$ one obtains again that the real roots exist in the energy range of $|E| \leqslant E_1 = 2\kappa$. A detailed study shows that this situation is similar to that for the isotropic system, with a marginal stability of the filter function.

The threshold disorder value reads now

$$\sigma_1(E) = 2\sqrt{\kappa}(1 - E^2/E_1^2)^{1/4}. \tag{43}$$

Therefore, one can see a monotonic decrease of both energy half-width $E_1 \propto \kappa$ (where delocalized states exist) and the threshold disorder $\sigma_1(E) \propto \sqrt{\kappa}$ in the limit $\kappa \to 0$ in the strong binding direction. For $\kappa = 0$ the phase of delocalized states loses its existence range, which corresponds to the well-known 1-D result.

3.3.2. The case of $P = +1$ (localized states)

Eq. (38) transforms into $w \geqslant 2\kappa + |E|$. Together with $w \geqslant 2$ one gets

$$w \geqslant u_0 = \max\{2, 2\kappa + |E|\}. \tag{44}$$

For the lack of disorder, $\sigma \to 0$, $w \to u_0$.

It is easy to find that at $|E| \leqslant E_2 = 2(1 - \kappa)$ the parameter $u_0 = 2$. Respectively, in this energy range for $\sigma = 0$ the Lyapunov exponent $\gamma(0, E) = 0$. At small σ the Lyapunov exponent is proportional to σ^2: $\gamma(\sigma, E) \sim \beta\sigma^2$. Analysis of Eq. (36) yields

$$\beta(E, \kappa) = \frac{1}{4}\left[\frac{1}{\sqrt{(E_{max} + E)(E_2 + E)}} + \frac{1}{\sqrt{(E_{max} - E)(E_2 - E)}}\right] \tag{45}$$

In the 1-D limit, $\kappa \to 0$, one gets $E_{max} = 2$, $E_2 = 2$, $|E| \leqslant 2$, respectively,

$$\beta(E, 0) = \frac{1}{4 - E^2}. \tag{46}$$

As one can expect, this corresponds to the exact 1-D result [17] (in the limit of small σ):

$$\gamma(\sigma, E) \sim \sigma^2/(4 - E^2). \tag{47}$$

In the energy range $E_2 \leqslant |E| \leqslant E_{max}$ $u_0 = 2\kappa + |E| \geqslant 2$. Respectively, the value

$$\gamma(0, E) = \cosh^{-1}\left(\kappa + \frac{|E|}{2}\right) \tag{48}$$

is non-zero.

Therefore, in the *isotropic* system with $\kappa = 1$ due to the presence of the phase transition Eq. (40) confirms completely the general conclusion, Eq. (25). The only exception is the band centre $E = 0$, where $\gamma(0, 0) = 0$

holds, and there is a correspondence with the main relation for the 1-D system—Eq. (24). As *anisotropy* arises, $\kappa > 0$, this point moves into the energy range $|E| \leqslant E_2$, where Eq. (24) holds. Moreover, within this range $\gamma(\sigma, E)$ can be expanded into a series in a small parameter σ^2, similar to the 1-D case. Outside this energy range Eq. (25) remains valid. As we have demonstrated, at $\kappa = 0$ our equations transform into those obtained earlier for the 1-D system [5,17].

3.4. Weak binding direction

Let us assume now $t_1 = \kappa$, $t_2 = 1$.

3.4.1. The case $P = -1$ (delocalized states)

In this case the following relations

$$\kappa|w| + |E| \leqslant 2 \tag{49}$$

and $0 \leqslant |w| \leqslant 2$ hold.

Real roots of equation $\mathscr{H}_-^{-1}(w) = 0$ exist in the parameter range $0 \leqslant |w| \leqslant \min\{2, (2 - |E|)/\kappa\}$ for energy $|E| \leqslant E_0 = 2$. The threshold disorder magnitude, where the delocalized states still exist, remains to be defined by Eq. (32):

$$\sigma_2(E) = 2\sqrt{\kappa}(1 - E^2/E_0^2)^{1/4}. \tag{50}$$

3.4.2. The case $P = +1$ (localized states)

Eq. (38) corresponds to

$$w \geqslant u_0 = \frac{(2 + |E|)}{\kappa}, \tag{51}$$

whereas the second condition $w \geqslant 2$ is fulfilled automatically. As $\sigma \to 0$, $w \to u_0 > 2$ holds. Therefore, Eq. (25) holds in the whole energy range.

3.5. Phase diagram

The delocalized state in the anisotropic system can be discussed only provided delocalization occurs simultaneously in two dimensions. It is easy to see that decisive points here are relations obtained for the strong binding direction. Therefore, the phase of delocalized states exists in the range $|E| \leqslant E_1 = 2\kappa$, provided for a given energy the disorder does not exceed the critical value, Eq. (43). A decrease of the anisotropy parameter κ leads to a monotonic decrease of both the energy range and the threshold disorder.

4. Conclusions

The phase of localized states in the anisotropic system can be characterized by *two* Lyapunov exponents $\gamma(\sigma, E)$ and, respectively, two localization lengths. The Lyapunov exponent in the strong binding direction is always smaller than that in the weak binding direction. In other words, the localization length in the strong binding direction is always larger than that in the perpendicular direction. These two localization lengths reveal different asymptotic behaviour as $\sigma \to 0$. In the limit of strong anisotropy, $\kappa \to 0$, only the localization length in the strong binding direction serves as an analogue to the single localization length in the 1-D system.

Acknowledgments

V.N.K. gratefully acknowledges the support of the Deutsche Forschungsgemeinschaft. The authors are indebted to E. Kotomin for detailed discussions and comments on the paper.

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